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Consistent scaling of multifractal measures: Multifractal spatial correlations

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There are a number of apparently disparate problems in multifractal scaling whose solutions have remained unclear, ranging from rather pathological cases where the standard Legendre transformations do not produce effective measures for the Hölder exponent and Hausdorff-Besicovitch dimension to the problem of describing the scaling of point-point correlation functions of moments of multifractal measures. We prove that an equivalent statement of multifractal scaling is the invariance of the generating functions of the scaling transformation. We show that the invariance of the generating functions is what allows the moment integrals to scale with simple power laws. We show that this definition can be successfully extended to cover the scaling of point-point correlation functions of moments of multifractal measures. Previous attempts to solve this problem have lead to non-scale-invariant behavior, presented as an inconsistency by Cates and Deutsch [Phys. Rev. A **35**, 4907 (1987)]. We propose that the invariance of generating functions under their own transformations is the central defining characteristic of scale invariance in multifractal scaling.

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I. INTRODUCTION

Multifractal scaling emerged as a theory holding great promise in solving some of the more intractable problems in fractal [1] growth [2]. It had immediate application to diffusion-limited growth [3–6] percolation and the random-resistor network [7–9], dynamical systems [10], and to the dissipation of energy in the eddy patterns of turbulent flow [11]. Many of the physically important parameters of these systems, such as the growth-sites distribution in diffusion-limited aggregation (DLA) [12, 3–6], or the energy distribution in turbulent flow [11], show a far more complicated scaling than the apparently simple fractal scaling of the geometry of the system. In this case, the size of the set of parameters p that scale $p \sim \epsilon^{\alpha}$ for some Hölder exponent α scales as $\epsilon^{f(\alpha)}$ for a Hausdorff-Besicovitch dimension $f(\alpha)$. Thus, for each of the α's in a continuum there is a fractal dimension. Since these physical parameters govern the dynamics of their systems, multifractal scaling offered a key to unraveling the physics behind complicated fractal systems.

Yet, there has emerged very little new insight into the behavior of physical systems from multifractal scaling. Even in DLA, one of the most important features, the spatial behavior of the screening, is unavailable in the straight growth-sites probabilities distribution. Further, the *ad hoc* nature of the standard formulation of multifractal scaling leaves something to be desired, and has motivated several attempts to formulate multifractals from more systematic approaches [13–15]. Definitions based solely on the extremization of the exponents which leads to the Legendre transformations also fail in pathological cases, which leads to the proliferation of yet more definitions on an *ad hoc* basis to cover these special cases [11,16]. Attempts to develop the consistent scaling of spatial correlations of multifractal measures based on the same kind of *ad hoc* formulation of multifractals also fails [17]. It seems clear that some form of systematic development that produces consistent-scale-invariant forms is necessary.

We note Coniglio [14] showed that transformation functions exhibiting power-law scaling necessarily have generating functions that are invariant under their own transformations. We approach the problem from the opposite side, as in Platt and Family [13], but we adapt the invariance of the generating functions under their own transformations as the definition of scale invariance. Since invariance of the generators of the transformations is both necessary and sufficient for multifractal scaling, it may be taken as a definition. We show that it necessarily follows that the scaling form of generating functions invariant under their own scaling necessarily demonstrates power-law scaling. We further show that it is that invariance that guarantees that the moments of the multifractal measures also demonstrate power-law scaling. Lastly, it is seen throughout the development that all the features of the standard formulation of multifractal scaling are preserved.

Since the foundation of the inconsistency that Cates and Deutsch derived was that the moment distributions did not scale commensurately with the distribution (that is, the scaling formulation was not scale invariant) [17], it seems clear that a formulation of the scaling of spatial correlations of moments of multifractal measures must follow from a foundation that guarantees consistency in the scaling transformations for the distributions and moments. Part of the problem is then to try to understand what about the nature of the scaling transformations caused them to not be consistent or scale invariant. We apply the same formalism developed above and show that the moments scale consistently with the distribution functions. We extract a form of scaling law similar to the one Cates and Deutsch sought, except that the result is scale invariant, and avoids the paradox.

As in the simple multifractal scaling example, it is the invariance of the generating functions that allows the moment correlation functions to scale consistently with the distribution function. We note that the invariance of the generating functions under their own transformation functions is a statement equivalent to other statements of scale invariance when applied to traditional multifractal scaling since it is both necessary and sufficient for power-law transformation functions. Further, this technique is extended to this problem more easily and systematically than the traditional approach. We conclude that the statement of the invariance of generating functions may be taken as a particularly useful definition of scale invariance in multifractal scaling. Further, the scaling exponents are now associated with generators of scaling transformation functions, which have meaning even where the extremization techniques that lead to Legendre transformations break down. This reduces the need for the proliferation of special definitions to cover pathological cases.

In Sec. II measure distribution functions are constructed from the moment definitions, scaling transformations are defined, and group conditions are imposed. The invariance of the generating functions under their own transformations is imposed to solve the group equations. In Sec. III the relationship between the scaling of the moments and the transformations that scale the distributions is explored. The moment integral is renormalized, and the relationship between the scaling of the moments and the distribution is established. In Sec. IV the technique is extended to the scaling of point-point correlation functions of moments of multifractal measures.

II. SCALING GROUP

In this section the distribution of multifractal measures is defined. Transformation functions are defined on the distribution. Group conditions are imposed on the transformation functions. The invariance of the generating functions is imposed on the transformation functions, and the group equations are solved.

A fractal is covered by a set Γ of balls $i\in\Gamma$ of size or norm ϵ. A measure $p_i(\epsilon)$ is defined on each $i\in\Gamma$. Moments at resolution ϵ may be defined

$$Z(q,\epsilon)=\sum_i p_i^q(\epsilon) \tag{1}$$

or

$$Z(q,\epsilon)=\int d^d\mathbf{x}\, p^q(\mathbf{x},\epsilon)\ . \tag{2}$$

In the standard formulation, the p_i's are said to scale as $p_i\sim\epsilon^{-\alpha}$ and the number of the p_i's that scale with α scales as $\epsilon^{-f(\alpha)}$. This presentation suffers from the difficulty that there is no identification between the p_i's at one ϵ and those at another ϵ since the coverings are both arbitrary and very dependent in a nontrivial way on ϵ [13,15]. The moments may be written in terms of a distribution by writing

$$Z(q,\epsilon)=\int d^d\mathbf{x}\int dp\, p^q\delta(p-p(\mathbf{x},\epsilon))\ ,$$

which leads to the expression of the moments

$$Z(q,\epsilon)=\int dp\, p^q n(p,\epsilon)\ , \tag{3}$$

in terms of the distribution function $n(p,\epsilon)$, where

$$n(p,\epsilon)=\int d^d\mathbf{x}\,\delta(p-p(\mathbf{x},\epsilon))\ . \tag{4}$$

A scaling transformation $T(s)$ may be defined which relates the distribution at length scale ϵ to the distribution at length scale ϵ/s. Such a transformation will have a form

$$n(p,\epsilon)dp\to T(s)n(p,\epsilon)dp$$
$$=w(p,\epsilon,s)n\left[pu(p,\epsilon,s),\frac{\epsilon}{s}\right]d(pu(p,\epsilon,s))\ . \tag{5}$$

The functions u and w multiply the variable p and the distribution n since the distributions are to be rescaled by the transformation. The nature of the group equations that emerge from this formulation must reflect the scaling nature of the transformation. The rescaling factor u is included in the differential to account for the Jacobian. The scaling constraint will be

$$T(s)n(p,\epsilon)dp=n(p,\epsilon)dp\ . \tag{6}$$

The group conditions to be imposed on the scaling transformations are identity and product. Namely, rescaling by a factor of $s=1$ should return the same distribution, whereas rescaling by a factor of s_1 followed by rescaling by a factor of s_2 should yield the same results as rescaling by the product s_1s_2 all at the same time. These may be expressed as

$$T(1)=1\ , \tag{7}$$

$$T(s_1s_2)=T(s_1)\circ T(s_2)\ . \tag{8}$$

These constraints may be expressed in terms of the transformation functions as

$$u(p,\epsilon,1)=1\ , \tag{9}$$

$$w(p,\epsilon,1)=1\ , \tag{10}$$

and

$$u(p,\epsilon,s_1 s_2)=u(p,\epsilon,s_1)u\left[pu(p,\epsilon,s_1),\frac{\epsilon}{s_1},s_2\right], \quad (11)$$

$$w(p,\epsilon,s_1 s_2)=w(p,\epsilon,s_1)w\left[pu(p,\epsilon,s_1),\frac{\epsilon}{s_1},s_2\right]. \quad (12)$$

The differential group equations may be obtained from the above by taking the partial $\partial/\partial s_1$, and then taking $s_1 \to 1$. The results are

$$s\frac{\partial u}{\partial s}=\Upsilon(p,\epsilon)u(p,\epsilon,s)+p\Upsilon(p,\epsilon)\frac{\partial u}{\partial p}-\epsilon\frac{\partial u}{\partial \epsilon}, \quad (13)$$

$$s\frac{\partial w}{\partial s}=W(p,\epsilon)w(p,\epsilon,s)+p\Upsilon(p,\epsilon)\frac{\partial w}{\partial p}-\epsilon\frac{\partial w}{\partial \epsilon}, \quad (14)$$

where

$$\Upsilon(p,\epsilon)=\frac{\partial u}{\partial s}(p,\epsilon,1), \quad (15)$$

$$W(p,\epsilon)=\frac{\partial w}{\partial s}(p,\epsilon,1). \quad (16)$$

The transformation functions may be integrated forward from $s=1$ to $s=1+b$ for $|b| \ll 1$ to yield

$$u(p,\epsilon,s)=1+\Upsilon(p,\epsilon)b \simeq s^{\Upsilon(p,\epsilon)}, \quad (17)$$

$$w(p,\epsilon,s)=1+W(p,\epsilon)b \simeq s^{W(p,\epsilon)}. \quad (18)$$

It becomes clear that the functions $\Upsilon(p,\epsilon)$ and $W(p,\epsilon)$ are generators of the infinitesimal transformations.

While these transformation functions already have a large number of constraints imposed upon them, they are still very general. A trivial transformation that would "scale" any transformation would be $u(p,\epsilon,s)=1$ and $w(p,\epsilon,s)=n(p,\epsilon)/n(p,\epsilon/s)$. It is clear that this definition satisfies all the group constraints, and yet will trivially scale any function arbitrarily. Scale invariance must be imposed upon the transformation functions and thus on the distributions. Coniglio [14] showed that the appropriate power-law form implied that the generating functions must be invariant under their own transformations. We will take that as a starting point in this development. In this case, the group equations are

$$T(s)\Upsilon(p,\epsilon)=\Upsilon\left[pu(p,\epsilon,s),\frac{\epsilon}{s}\right]=\Upsilon(p,\epsilon), \quad (19)$$

$$T(s)W(p,\epsilon)=W\left[pu(p,\epsilon,s),\frac{\epsilon}{s}\right]=W(p,\epsilon). \quad (20)$$

The differential form of the above equations is

$$-\epsilon\frac{\partial \Upsilon}{\partial \epsilon}+p\Upsilon\frac{\partial \Upsilon}{\partial p}=0, \quad (21)$$

$$-\epsilon\frac{\partial W}{\partial \epsilon}+p\Upsilon\frac{\partial W}{\partial p}=0. \quad (22)$$

The above equations may be integrated along the characteristics defined by

$$\epsilon\frac{dp(\epsilon)}{d\epsilon}=-p\Upsilon(p(\epsilon),\epsilon). \quad (23)$$

Substituting this into the partial-differential equations (PDE's) yields

$$\frac{d\Upsilon(p(\epsilon),\epsilon)}{d\epsilon}=0, \quad (24)$$

$$\frac{dW(p(\epsilon),\epsilon)}{d\epsilon}=0, \quad (25)$$

which implies that the generators are constants along the characteristics. Substituting this back into the characteristic equation, it follows that

$$\frac{d\ln p}{d\ln\epsilon}=-\Upsilon(p(\epsilon),\epsilon) \quad (26)$$

or

$$p(\epsilon)=p(1)\epsilon^{-\Upsilon(p(1),1)}. \quad (27)$$

The differential group equations may be solved along the same characteristics. Substituting the equation for the characteristic into the equations for u and w yields

$$\frac{\partial u(p(\epsilon),\epsilon,s)}{\partial \epsilon}+s\frac{\partial u(p(\epsilon),\epsilon,s)}{\partial s}-\Upsilon u=0,$$

$$\epsilon\frac{\partial w(p(\epsilon),\epsilon,s)}{\partial \epsilon}+s\frac{\partial w(p(\epsilon),\epsilon,s)}{\partial s}-Ww=0,$$

which may be rewritten

$$\left[\epsilon\frac{\partial}{\partial\epsilon}+s\frac{\partial}{\partial s}\right](\ln u-\Upsilon\ln s)=0,$$

$$\left[\epsilon\frac{\partial}{\partial\epsilon}+s\frac{\partial}{\partial s}\right](\ln w-W\ln s)=0.$$

These equations have solutions of the form

$$\ln u-\Upsilon\ln s=\phi_1(\ln\epsilon-\ln s),$$

$$\ln w-W\ln s=\phi_2(\ln\epsilon-\ln s)$$

for some ϕ_1 and ϕ_2. This function may be determined from the boundary condition at $s=1$. Then

$$\phi_1(\ln\epsilon)=0,$$

$$\phi_2(\ln\epsilon)=0,$$

which, when substituted back into the solution, yields

$$u(p,\epsilon,s)=s^{-\Upsilon(p,\epsilon)}, \quad (28)$$

$$w(p,\epsilon,s)=s^{-W(p,\epsilon)}. \quad (29)$$

From the preceding, it becomes clear that the generators Υ and W may be associated with the Hölder singularity exponents α and the Hausdorff-Besicovitch dimensions $f(\alpha)$, respectively. However, the identification is not between measures on a ball $i \in \Gamma$ as ϵ changes, but rather a characteristic curve that describes scale invariance within a family of distributions related through scale invariance to each other as ϵ changes. This further associates these exponents and dimensions with the scaling generators for a distribution. These generators have a definition and a role even in the pathological case where the Legendre-transformation definition does not extract the extremum along the exponent under the moment integral, which is the traditional way of operationally defining the Hölder exponents and the Hausdorff-Besicovitch dimensions.

III. SCALING OF THE MOMENT INTEGRALS

In this section we apply the scaling transformation developed in Sec. II to the moment integrals. We show that the invariance of the generators under their own transformations is the central reason why the moments reveal scale-invariant power-law behavior.

The scaling of the moment integral $Z(q,\epsilon)$ has the form

$$T(s)Z(q,\epsilon)=Z\left[q,\frac{\epsilon}{s}\right]s^{(q-1)D(q)}$$
$$=\int p^{q}s^{W(p,\epsilon)}n\left[ps^{\Upsilon(p,\epsilon)},\frac{\epsilon}{s}\right]d\{ps^{\Upsilon(p,\epsilon)}\}\ . \tag{30}$$

The substitution $y=ps^{\Upsilon(p,\epsilon)}=ps^{\Upsilon(y,\epsilon/s)}$ or $p=ys^{-\Upsilon(y,\epsilon/s)}$ yields

$$T(s)Z(q,\epsilon)=Z\left[q,\frac{\epsilon}{s}\right]s^{(q-1)D(q)}$$
$$=\int\{ys^{-\Upsilon(y,\epsilon/s)}\}^{q}s^{W(y,\epsilon/s)}n\left[y,\frac{\epsilon}{s}\right]dy$$
$$=\int s^{W(y,\epsilon/s)-q\Upsilon(y,\epsilon/s)}y^{q}n\left[y,\frac{\epsilon}{s}\right]dy\ . \tag{31}$$

Along the contours defined by

$$\alpha=\Upsilon(p,\epsilon)=\Upsilon(y,\epsilon/s)\ , \tag{32}$$

$$f(\alpha)=W(p,\epsilon)=W(y,\epsilon/s)\ , \tag{33}$$

where $p=p(\alpha,\epsilon)$ and $y=p(\alpha,\epsilon/s)$, limits for both large s and $s=1+b$ for $|b|\ll 1$ may be applied. For the large-s limit, it follows that the integral will be dominated by those values of α such that the exponent of s will be maximum. It follows that the standard Legendre-transformations result, namely

$$(q-1)D(q)=f(\alpha(q))-q\alpha(q)\ , \tag{34}$$

$$\frac{d\{(q-1)D(q)\}}{dq}=-\alpha(q)\ . \tag{35}$$

The values of $p(\alpha,\epsilon)$ may be obtained by examining the integral with small s. In that case, the integral will be dominated by the extremum of the distribution, namely

$$\left[\frac{\partial p^{q}n(p,\epsilon)}{\partial p}\right]_{p=p(\alpha(q),\epsilon)}=0\ . \tag{36}$$

The above arguments show clearly that the power-law form can be extracted from under the integral only because (1) the transformation functions are power laws, and (2) the exponents remain constant along the contours. It is only by parametrizing the transformation function by the contours that the association between the dimensions $D(q)$ of the moments and the generating functions can be constructed. Without this invariance, the same kind of paradox derived by Cates and Deutsch [17] would emerge here. A corollary to that is the suggestion that this technique may be extended to guarantee the invariance of scaling transformations in the more complicated situations where the traditional *ad hoc* formulation fails.

IV. SCALING OF MULTIFRACTAL SPATIAL CORRELATIONS

One of the strongest demonstrations of a formalism is to show that it solves a problem that has persisted in the field for a long time. An example of this is the Cates and Deutsch incompatibility [17]. In their case, they were interested in deriving the scaling form for spatial correlations of moments of multifractal measures both from a blob technique, and via a hierarchy of singularities, trying to extend the arguments of Halsey, Meakin, and Procaccia [4]. They found that the various scaling parameters from the blob arguments did not extremize in the Legendre-transformation space, imposed by the hierarchy of singularities picture, in a way that would separate the exponents of the various scaling parameters. This suggests that either the hierarchy picture, or the blob picture, or both are not scale invariant.

From the perspective of this study, scale invariance of the transformation functions is the starting point. The problem is that the scaling Ansätze derived by Cates and Deutsch are not scale invariant; they do not form a product group with generating functions that satisfy the constraints required to guarantee the power-law scaling of the moment functions. It is not clear from the structure of the arguments presented in Cates and Deutsch exactly where the noninvariant transformations are introduced, since elements from both the hierarchical picture and the blob picture are present in the integral to be extremized.

At the same time, reconciling both the scaling of the moments with the scaling of the distribution is important. In the context of Cates and Deutsch, the scaling of the spatial correlations of the moments was handled by the blob arguments. The scaling of the distribution was handled by the hierarchic picture. Cates and Deutsch concluded that the two pictures are incompatible. However, the problem is more significant than that. Since the correlation functions may be expressed in terms of integrals over the distribution, it is necessary to resolve the scaling of both sides of the equation. It is paradoxical that the correlation of moments would be scale invariant and the distribution would not be or vice versa. We will identify this inconsistency as a paradox.

In this section the problem will be approached using the scaling-group techniques presented in Secs. II and III. It does not assume anything about blobs or hierarchical singularities. Instead, it assumes only that there are scaling transformations and proceeds from that point to derive relationships between them. It will be seen that the scaling groups produce results that guarantee scale invariance, and produce consistent power-law scaling forms for the moment correlation functions.

First, the moment correlation function is defined

$$C(q_1,q_2,\xi,\epsilon)=\frac{1}{N}\left\langle \int d^d\mathbf{x}\, p^{q_1}(\mathbf{x},\epsilon)p^{q_2}(\mathbf{x}+\xi,\epsilon)\right\rangle_{\Omega(\hat{\xi})} . \tag{37}$$

This requires that, for any one covering on balls of size ϵ, that the products of the p's separated by ξ be averaged. In order to explore the spatial dependence of multifractal measures, a distribution function must be constructed that reflects the spatial dependence of the measures. However, again, there will be no identity between balls at one ϵ with those at another ϵ. It will be necessary to examine the behavior only of the distribution of measures at different length scales.

As before, the average can be expressed as an integral

$$C(q_1,q_2,\xi,\epsilon)=\frac{1}{N\Omega_d}\oint d\Omega_d(\hat{\xi})\int d^d\mathbf{x}\, p^{q_1}(\mathbf{x},\epsilon)p^{q_2}(\mathbf{x}+\xi,\epsilon)\rho(\mathbf{x},\epsilon)\rho(\mathbf{x}+\xi,\epsilon) , \tag{38}$$

which can be integrated out

$$\begin{aligned}C(q_1,q_2,\xi,\epsilon)&=\frac{1}{N\Omega_d}\oint d\Omega_d(\hat{\xi})\int d^d\mathbf{x}\, p^{q_1}(\mathbf{x},\epsilon)p^{q_2}(\mathbf{x}+\xi,\epsilon)\rho(\mathbf{x},\epsilon)\rho(\mathbf{x}+\xi,\epsilon)\\&=\frac{1}{N\Omega_d}\oint d\Omega_d(\hat{\xi})\int d^d\mathbf{x}\int dp_1\int dp_2 p_1^{q_1}p_2^{q_2}\,\delta(p_1-p(\mathbf{x},\epsilon))\delta(p_2-p(\mathbf{x}+\xi,\epsilon))\rho(\mathbf{x},\epsilon)\rho(\mathbf{x}+\xi,\epsilon)\\&=\int dp_1\int dp_2 p_1^{q_1}p_2^{q_2}\frac{1}{N\Omega_d}\oint d\Omega_d(\hat{\xi})\int d^d\mathbf{x}\,\delta(p_1-p(\mathbf{x},\epsilon))\delta(p_2-p(\mathbf{x}+\xi,\epsilon))\rho(\mathbf{x},\epsilon)\rho(\mathbf{x}+\xi,\epsilon) .\end{aligned}$$

The distribution function may be identified as

$$n(p_1,p_2,\xi,\epsilon)dp_1dp_2=\frac{1}{N\Omega_d}\oint d\Omega_d(\hat{\xi})\int d^d\mathbf{x}\,\delta(p_1-p(\mathbf{x},\epsilon))\delta(p_2-p(\mathbf{x}+\xi,\epsilon))\rho(\mathbf{x},\epsilon)\rho(\mathbf{x}+\xi,\epsilon)dp_1dp_2 . \tag{39}$$

The moment correlation function is then

$$C(q_1,q_2,\xi,\epsilon)=\int dp_1\int dp_2 p_1^{q_1}p_2^{q_2}n(p_1,p_2,\xi,\epsilon) . \tag{40}$$

Since there is an integral relationship between C and n, it follows that the scaling of each of these elements must be compatible. Besides representing two pictures of multifractals, the Cates and Deutsch arguments reflect the scaling of each side of the above equation. The blob arguments describe the scaling of the left-hand side, and the hierarchical picture describes the scaling of the right-hand side. It is not enough just to invalidate one picture or the other. The formulation must be consistent on both sides of the equation. The fact that they derived an inconsistency reflects a very serious and fundamental problem with the formulation.

The scaling form for this distribution is

$$T(s)n(p_1,p_2,\xi,\epsilon)dp_1dp_2=wn\left[p_1u_1,p_2u_2,\xi,\frac{\epsilon}{s}\right]J\left[\frac{p_1u_1,p_2u_2}{p_1,p_2}\right]dp_1dp_2 , \tag{41}$$

where w, u_1, and u_2 are all functions of (p_1,p_2,ξ,ϵ,s). These must also satisfy the group equations, which in functional form are

$$u_1(p_1,p_2,\xi,\epsilon,1)=1 , \tag{42}$$

$$u_2(p_1,p_2,\xi,\epsilon,1)=1 , \tag{43}$$

$$w(p_1,p_2,\xi,\epsilon,1)=1 , \tag{44}$$

$$u_1(p_1,p_2,\xi,\epsilon,s_1s_2)=u_1(p_1,p_2,\xi,\epsilon,s_1)u_1\left[p_1u_1(p_1,p_2,\xi,\epsilon,s_1) ,p_2u_2(p_1,p_2,\xi,\epsilon,s_1),\xi,\frac{\epsilon}{s_1},s_2\right] , \tag{45}$$

$$u_2(p_1,p_2,\xi,\epsilon,s_1s_2)=u_2(p_1,p_2,\xi,\epsilon,s_1)u_2\left[p_1u_1(p_1,p_2,\xi,\epsilon,s_1),p_2u_2(p_1,p_2,\xi,\epsilon,s_1),\xi,\frac{\epsilon}{s_1},s_2\right] , \tag{46}$$

$$w(p_1,p_2,\xi,\epsilon,s_1s_2)=w(p_1,p_2,\xi,\epsilon,s_1)w\left[p_1u_1(p_1,p_2,\xi,\epsilon,s_1) ,p_2u_2(p_1,p_2,\xi,\epsilon,s_1),\xi,\frac{\epsilon}{s_1},s_2\right] . \tag{47}$$

The differential form of the group equations is

$$s\frac{\partial u_1}{\partial s}=\Upsilon_1 u_1+p_1\Upsilon_1\frac{\partial u_1}{\partial p_1}+p_2\Upsilon_2\frac{\partial u_1}{\partial p_2}-\epsilon\frac{\partial u_1}{\partial \epsilon}\ , \tag{48}$$

$$s\frac{\partial u_2}{\partial s}=\Upsilon_2 u_2+p_1\Upsilon_1\frac{\partial u_2}{\partial p_1}+p_2\Upsilon_2\frac{\partial u_2}{\partial p_2}-\epsilon\frac{\partial u_2}{\partial \epsilon}\ , \tag{49}$$

$$s\frac{\partial w}{\partial s}=Ww+p_1\Upsilon_1\frac{\partial w}{\partial p_1}+p_2\Upsilon_2\frac{\partial w}{\partial p_2}-\epsilon\frac{\partial w}{\partial \epsilon}\ , \tag{50}$$

where

$$\Upsilon_1=\frac{\partial u_1}{\partial s}(p_1,p_2,\xi,\epsilon,1)\ , \tag{51}$$

$$\Upsilon_2=\frac{\partial u_2}{\partial s}(p_1,p_2,\xi,\epsilon,1)\ , \tag{52}$$

$$W=\frac{\partial w}{\partial s}(p_1,p_2,\xi,\epsilon,1)\ . \tag{53}$$

The scale-invariance conditions, analogous to those developed in Sec. II, expressed in functional form are

$$\Upsilon_1(p_1,p_2,\xi,\epsilon)=\Upsilon_1\left[p_1u_1,p_2u_2,\xi,\frac{\epsilon}{s}\right]\ , \tag{54}$$

$$\Upsilon_2(p_1,p_2,\xi,\epsilon)=\Upsilon_2\left[p_1u_1,p_2u_2,\xi,\frac{\epsilon}{s}\right]\ , \tag{55}$$

$$W(p_1,p_2,\xi,\epsilon)=W\left[p_1u_1,p_2u_2,\xi,\frac{\epsilon}{s}\right]\ , \tag{56}$$

which have the differential form

$$p_1\Upsilon\frac{\partial \Upsilon_1}{\partial p_1}+p_2\Upsilon_2\frac{\partial \Upsilon_1}{\partial p_2}-\epsilon\frac{\partial \Upsilon_1}{\partial \epsilon}=0\ , \tag{57}$$

$$p_1\Upsilon_1\frac{\partial \Upsilon_2}{\partial p_1}+p_2\Upsilon_2\frac{\partial \Upsilon_2}{\partial p_2}-\epsilon\frac{\partial \Upsilon_2}{\partial \epsilon}=0\ , \tag{58}$$

$$p_1\Upsilon_1\frac{\partial W}{\partial p_1}+p_2\Upsilon_2\frac{\partial W}{\partial p_2}-\epsilon\frac{\partial W}{\partial \epsilon}=0\ . \tag{59}$$

These equations may be solved using the method of characteristics, by choosing the ξ-dependent characteristics

$$\epsilon\frac{dp_1(\epsilon)}{d\epsilon}=-p_1\Upsilon_1(p_1(\epsilon),p_2(\epsilon),\xi,\epsilon)\ , \tag{60}$$

$$\epsilon\frac{dp_2(\epsilon)}{d\epsilon}=-p_2\Upsilon_2(p_1(\epsilon),p_2(\epsilon),\xi,\epsilon)\ . \tag{61}$$

Along these characteristics, the partial-differential equations reduce to

$$-\epsilon\frac{d\Upsilon_1(p_1(\epsilon),p_2(\epsilon),\xi,\epsilon)}{d\epsilon}=0\ , \tag{62}$$

$$-\epsilon\frac{d\Upsilon_2(p_1(\epsilon),p_2(\epsilon),\xi,\epsilon)}{d\epsilon}=0\ , \tag{63}$$

$$-\epsilon\frac{dW(p_1(\epsilon),p_2(\epsilon),\xi,\epsilon)}{d\epsilon}=0\ . \tag{64}$$

So, along the characteristics, Υ_1, Υ_2, and W are all constant. The form of the differential equations for u_1, u_2, and w along the characteristics $p_1=p_1(\epsilon)$ and $p_2=p_2(\epsilon)$ are

$$s\frac{\partial u_1}{\partial s}+\epsilon\frac{\partial u_1}{\partial \epsilon}-\Upsilon_1u_1=0\ , \tag{65}$$

$$s\frac{\partial u_2}{\partial s}+\epsilon\frac{\partial u_2}{\partial \epsilon}-\Upsilon_2u_2=0\ , \tag{66}$$

$$s\frac{\partial w}{\partial s}+\epsilon\frac{\partial w}{\partial \epsilon}-Ww=0\ . \tag{67}$$

These may be solved in the same way that the simpler equations were solved for the simple multifractal case. The solutions are

$$u_1(p_1,p_2,\xi,\epsilon,s)=s^{\Upsilon_1(p_1,p_2,\xi,\epsilon)}\ , \tag{68}$$

$$u_2(p_1,p_2,\xi,\epsilon,s)=s^{\Upsilon_2(p_1,p_2,\xi,\epsilon)}\ , \tag{69}$$

$$w(p_1,p_2,\xi,\epsilon,s)=s^{W(p_1,p_2,\xi,\epsilon)}\ . \tag{70}$$

Next, the scaling of the correlation moments must be derived. First, the moments may be expressed in terms of n by writing

$$C(q_1,q_2,\xi,\epsilon)=\int dp_1\int dp_2 p_1^{q_1}p_2^{q_2}n(p_1,p_2,\xi,\epsilon)\ .$$

The scaling form for this kind of expression will be

$$C(q_1,q_2,\xi,\epsilon)=C\left[q_1,q_2\xi,\frac{\epsilon}{s}\right]s^{-D(q_1,q_2,\xi)}\ . \tag{71}$$

Substituting the scaling forms into the integral yields

$$C(q_1,q_2,\xi,\epsilon)=\int dp_1\int dp_2 J\left[\frac{p_1u_1,p_2u_2}{p_1,p_2}\right]p_1^{q_1}p_2^{q_2}wn\left[p_1u_1,p_2u_2,\xi,\frac{\epsilon}{s}\right]\ . \tag{72}$$

Making the change of variable $y_1=p_1s^{\Upsilon_1(p_1,p_2,\xi,\epsilon)}$ and $y_2=p_2s^{\Upsilon_2(p_1,p_2,\xi,\epsilon)}$, the integral becomes

$$C(q_1,q_2,\xi,\epsilon)=\int dy_1\int dy_2 s^{W(y_1,y_2,\xi,\epsilon/s)-q_1\Upsilon_1-q_2\Upsilon_2}y_1^{q_1}y_2^{q_2}n\left[p_1u_1,p_2u_2,\xi,\frac{\epsilon}{s}\right]\ , \tag{73}$$

where the Jacobian simplified to unity by the substitution. For $s \sim O(1/\epsilon)$, the integral will be dominated by the extremum of the exponent of s. Defining the characteristics

$$\Upsilon_1(y_1(\alpha_1,\alpha_2,\xi,\epsilon),y_2(\alpha_1,\alpha_2,\xi,\epsilon),\xi,\epsilon)=\alpha_1 \ , \tag{74}$$

$$\Upsilon_2(y_1(\alpha_1,\alpha_2,\xi,\epsilon),y_2(\alpha_1,\alpha_2,\xi,\epsilon),\xi,\epsilon)=\alpha_2 \ , \tag{75}$$

with the spectrum function defined

$$f(\alpha_1,\alpha_2,\xi)=W(y_1(\alpha_1,\alpha_2,\xi,\epsilon),y_2(\alpha_1,\alpha_2,\xi,\epsilon),\xi,\epsilon) \ , \tag{76}$$

which will also be constant along the characteristic contour since Υ_1, Υ_2, and W are ϵ-independent along the characteristic contour. The exponent extremizes where

$$\frac{\partial f(\alpha_1,\alpha_2,\xi)}{\partial \alpha_1}(\alpha_1(q_1,q_2,\xi),\alpha_2(q_1,q_2,\xi),\xi)=q_1 \ , \tag{77}$$

$$\frac{\partial f(\alpha_1,\alpha_2,\xi)}{\partial \alpha_2}(\alpha_1(q_1,q_2),\alpha_2(q_1,q_2),\xi)=q_2 \ . \tag{78}$$

Then, it follows

$$D(q_1,q_2,\xi)=q_1\alpha_1(q_1,q_2,\xi)+q_2\alpha_2(q_1,q_2,\xi) -f(\alpha_1(q_1,q_2,\xi),\alpha_2(q_1,q_2,\xi),\xi) \ , \tag{79}$$

$$\frac{\partial D(q_1,q_2,\xi)}{\partial q_1}=\alpha_1(q_1,q_2,\xi) \ , \tag{80}$$

$$\frac{\partial D(q_1,q_2,\xi)}{\partial q_2}=\alpha_2(q_1,q_2,\xi) \ . \tag{81}$$

The values of p_1 and p_2 associated with each α must be determined to reconstruct the Υ's and W. This information may be obtained by examining the integral in the limit of $s \sim 1$. The integral will be dominated by the extremum of $y_1^{q_1}y_2^{q_2}n(y_1,y_2,\xi,\epsilon/s)$. These occur where

$$\left[\frac{\partial y_1^{q_1}y_2^{q_2}n(y_1,y_2,\xi,\epsilon/s)}{\partial y_1}\right]_{(y_1,y_2)=(y_1(q_1,q_2,\xi,\epsilon/s),y_2(q_1,q_2,\xi,\epsilon/s))}=0 \ , \tag{82}$$

$$\left[\frac{\partial y_1^{q_1}y_2^{q_2}n(y_1,y_2,\xi,\epsilon/s)}{\partial y_2}\right]_{(y_1,y_2)=(y_1(q_1,q_2,\xi,\epsilon/s),y_2(q_1,q_2,\xi,\epsilon/s))}=0 \ . \tag{83}$$

Scale invariance is guaranteed in the above formulation. The ϵ dependence is completely removed from being the kind of problem it posed in the Cates and Deutsch paradox [17]. Further, the details of the ξ dependence are not dictated by the behavior of the scaling of the p's on the balls as ϵ changes, but is left as an open function dependent on the other details of the system being considered, which can have a nontrivial behavior completely separate and distinct from the behavior of some otherwise arbitrary measure on the system.

V. CONCLUSIONS

One of the dominant features of multifractal scaling has been the *ad hoc* nature of the constructions of the scaling forms. The distributions of the multifractal measures were always present at least implicitly, yet few studies have focused on the kind of detail that would be required to obtain scaling collapse of the distributions. Further, when pathological situations arose in the standard techniques of defining or measuring the Hölder exponents and Hausdorff-Besicovitch dimensions, the tendency has been to proliferate more dimensions. Lastly, it has been impossible to successfully extend multifractal scaling to more complicated situations by following this *ad hoc* approach. The problems that seem to emerge with greater or lesser clarity in all these situations involved the problem of the definition of scale invariance. The task at hand more and more clearly became one of seeking that definition.

In this development, the association between Hölder exponents and Hausdorff-Besicovitch dimensions and the generating functions becomes very clear. The exponents and dimensions are not trivial enough to define in an *ad hoc* way, but perform the role of describing the scaling of the multifractal distributions. They are the functions that would be necessary to produce a scaling collapse of the distribution function of the multifractal measures. Further, we show that the invariance of the generating functions along the characteristics is necessary if the Hölder exponents and Hausdorff-Besicovitch dimensions are to even be definable, and for the moment integrals to be scale invariant. We show that the invariance of the generating functions is mathematically equivalent to previous definitions of scale invariance in the context of multifractal scaling. Lastly, by successfully extending this definition to the problem of correlations of moments of multifractal measures, thus solving a long standing problem, we show that the usefulness of the definition goes beyond the context of simple multifractal scaling. From these results we conclude that the invariance of the generating functions under their own transformations of the multifractal-scaling-transformation functions is a fundamental principle in defining multifractal scale invariance.

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